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Recent developments in the digital approach of symbolic dynamics

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Abstract

Purpose – The purpose of this paper is to explore new mathematical results to advance the understanding of the picture of a chaotic unimodal map.

Design/methodology/approach – Ever since Poicare, deterministic chaos is ultimately connected with exponential divergence of nearby trajectories, unpredictability and erratic behaviour. Here, the authors propose an alternative approach in terms of complexity theory and transcendence.

Findings – In this paper, the authors were able to reproduce previous results easily, due to new theorems.

Originality/value – The paper updates previous results and proposes a more complete understanding of the phenomenon of deterministic chaos, also making possible connections with number theory, combinatorics and possibly quantum mechanics, as in quantum mechanics there does not exist the notion on nearby trajectories.

Keywords Chaos theory, Complexity theory, Determinants

Paper type Conceptual paper

The discovery that simple deterministic systems can show a vast richness of behaviours in response to variations of initial conditions and/or control parameters, has been at the origin of an intense interdisciplinary activity during the last two decades (Schroder, 1991). One of the outcomes of this effort has been the realization that for an appropriate description of such complex systems, one needs to resort to a probabilistic approach (Nicolis and Gaspard, 1994). Now, once one leaves the description in terms of trajectories, a basic question that must be dealt with concerns the amount of information necessary to follow the evolution of the system in the course of time. One of the approaches developed in this context is coarse graining, whereby a complex system is viewed as an information generator producing messages constituted of a discrete set of symbols defined by partitioning the full continuous phase space into a finite number of cells. We refer to such a description as "symbolic dynamics". One of its additional merits is to provide also a link between dynamical

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systems, information theory and cognitive processes (Schroder, 1991; Nicolis and Gaspard, 1994).

On the other hand, one may equally well follow the evolution of the system by assigning digits rather than abstract symbols to the cells of the partition. In this respect, symbolic dynamics leads to a "digital approach". This approach is in turn intimately connected to combinatorics and number theory and one can take profit from new theorems and advances in these fields.

In a previous paper (Karamanos, 2001), we established a connexion between dynamical systems and the property of their symbolic sequences to be algebraic irrational or transcendental. We restricted ourselves to the class of dynamical systems amenable to a one-to-one dimensional recurrence on the interval, the so-called unimodal maps.

In particular, ever since Poincare, chaos is viewed through its exponential sensitivity to initial conditions and erratic behaviour. In Karamanos (2001), we have attempted an alternative description of the same phenomenon in terms of algebraic properties of the numbers corresponding to the symbolic dynamics of the generating partition.

To this end, we have introduced for the Feigenbaum point, a number that we called "k", whose the binary expansion is generated by a finite automaton of 2-states ("2-automatic"), and can in an equivalent manner be generated by the algorithm of Metropolis *et al.* (MSS algorithm) (Metropolis *et al.*, 1973; Derrida *et al.*, 1978) or (in view of a theorem by Cobham (1972)) be viewed as the fixed point of the morphism g defined as:

$$g(0) = [11], \quad g(1) = [10],$$

starting with "1", that is:

$$\kappa = 0.101110101011101\ldots(\text{base } 2)$$

or:

$$\kappa = 0.729427\ldots(\text{base } 10).$$

This is a kind of "superuniversal constant", as it is valid for the Feigenbaum attractor of any unimodal map.

In Karamanos (2001), we have been based in the following theorem in order to prove rigorously the transcendence of "κ".

Theorem 1. (Allouche and Zamboni, 1998) Let x be a positive real number whose binary expansion is a fixed point of a morphism on the alphabet $\{0,1\}$. If the morphism is either of constant length ≥ 2 or primitive, then the number x is either rational or transcendental.

Recently, Adamczewski and Bugeaud (2007) have proved the conjecture of Loxton and van der Poorten in its more general form (that is, that irrational automatic numbers are transcendental). More specifically, they have shown that.

Theorem 2. (Adamczewski and Bugeaud, 2007) Let $b \geq 2$ be an integer. The b-adic expansion of any irrational algebraic number cannot be generated by a finite automaton. In other words, irrational automatic numbers are transcendental.

From this theorem, the transcendence of the number "κ" follows in a straightforward manner, as its binary expansion is generated by a finite automaton with two states.

Furthermore, Adamczewski and Bugeaud (2007) have proved one other important theorem that implies immediately the transcendence of the numbers defined for the accumulation points of the $m \cdot 2^{k}$ superstable cycles. (We do not enter in so much detail here.)

Theorem 3. (Adamczewski and Bugeaud, 2007) Binary algebraic irrational numbers cannot be generated by a morphism.

We thus find in a different context (this of automata and turing machines) the results announced in Karamanos (2000, 2001). This should validate further our philosophical positions.

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